S.T. Yau College Student Mathematics Contest
Applied and Computation Math (Group Final)
June 10, 2023

Problem 1: LU factorization

Let $A \in \mathbb{R}^n$ be a real tridiagonal matrix

$$A = \begin{bmatrix} \alpha_1 & \gamma_2 & & & \\ \beta_1 & \alpha_2 & \gamma_3 & & \\ & \beta_2 & \alpha_3 & \ddots & \\ & & \ddots & \ddots & \gamma_n \\ & & & \beta_{n-1} & \alpha_n \end{bmatrix}.$$

Consider applying the LU factorization with partial pivoting to this A to get $LU = PA$. The algorithm ("Algorithm 3") proceeds as follows, where L and U are stored in the lower and upper parts of A respectively:

Algorithm 3: LU decomposition with partial pivoting

for $k = 1, \ldots, n-1$ **do**
  Find the smallest index L such that $|A(l,k)| = \max_{k \le i \le n} |A(i,k)|$;
  Swap $A(k, 1:n)$ and $A(l, 1:n)$ and record the pair (k, l);
  for $i = k+1, \ldots, n$ **do**
    $A(i,k) = A(i,k)/A(k,k)$;
    $A(i, k+1:n) = A(i, k+1:n) - A(i,k) * A(k, k+1:n)$

(i). Let $a^* = \max_{i,j} |a_{ij}|$. Show that $|u_{ii}| \le 2a^*$, $\forall i$, and $|u_{ij}| \le a^*$, $\forall j > i$. Then conclude that the growth factor $\rho = \frac{\max_{i,j} |u_{ij}|}{\max_{i,j} |a_{ij}|} \le 2$.

(ii). Assume that $|\alpha_1| \ge |\beta_1|$, $|\alpha_n| \ge |\gamma_n|$, and $|\alpha_i| \ge |\beta_i| + |\gamma_i|$, $i = 2, \cdots, n-1$, i.e., A is column diagonally dominant. Show that the LU factorization with or without partial pivoting are step-wise equivalent for A. In other words, for the algorithm with partial pivoting, no actual pivoting happens throughout the process.

Problem 2: Deformation to Legendre transform

Given a strictly convex function $f : \mathbb{R}^n \to \mathbb{R}$ that is at least twice differentiable, its *Legendre transform* $f^* : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$f^*(u) = \sup_x \{\langle x, u \rangle - f(x)\},$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^n$ and $\langle x, u \rangle$ is the bilinear form. Recall that ∇f and ∇f^* are inverse functions of each other, and that $Hessf$ and $Hessf^*$ are inverse matrices of each other. Here ∇f and $Hessf$ denote, respectively, the first and second derivative of the function f:

$$\nabla f = [\frac{\partial f}{\partial x^1}, \cdots, \frac{\partial f}{\partial x^n}], \quad Hess_{ij} f = \frac{\partial f}{\partial x^i \partial x^j}\ .$$

(i) For any *fixed* real number λ in an open neighborhood of 0, the λ*-deformed Legendre transform* of f is defined by

$$f^{(\lambda)}(u) = \sup_x \left\{ \frac{1}{\lambda} \log\left(1 + \lambda\langle x, u\rangle\right) - f(x) \right\},$$

so that $\lim_{\lambda\to 0} f^{(\lambda)}(u) = f^*(u)$. Show that

$$f(x) + f^{(\lambda)}(u^{(\lambda)}) = \frac{1}{\lambda} \log\left(1 + \lambda\langle x, u^{(\lambda)}\rangle\right),$$

where

$$u^{(\lambda)} = \frac{\nabla f(x)}{1 - \lambda\langle x, \nabla f(x)\rangle}.$$

The righthand side of the above can be called the λ-gradient of f.

(ii) Define $x^{(\lambda)} \equiv x e^{-\lambda f(x)}$, and define the function $g^{(\lambda)}$ by

$$g^{(\lambda)}(x^{(\lambda)}) = \frac{1}{\lambda}\left(1 - e^{-\lambda f(x)}\right).$$

Show

$$u^{(\lambda)} = \nabla g^{(\lambda)}(x^{(\lambda)})$$

by explicitly evaluating the Jacobian of the transform $x \longleftrightarrow x^{(\lambda)}$.

(iii) Calculate $(f^{(\lambda)})^{(\lambda)}$ and state whether $(f^{(\lambda)})^{(\lambda)} = f$ holds.

Problem 3: Numerical PDE

Let K and $\widehat{K}$ be two affine-equivalent bounded open subsets of $\mathbb{R}^d$, that is, there is a bijiective affine mapping $F : \widehat{K} \to K$ defined by $F(\widehat{x}) = B\widehat{x} + b$, where B is a nonsingular matrix and $b \in \mathbb{R}^d$.

(i). Let $v(x) \in H^m(K)$ and $\widehat{v}(\widehat{x}) = v(F(\widehat{x})) \in H^m(\widehat{K})$. Prove that

$$|\widehat{v}|_{H^m(\widehat{K})} \le C\|B\|^m |\det(B)|^{-\frac{1}{2}} |v|_{H^m(K)}$$

and

$$|v|_{H^m(K)} \le C\|B^{-1}\|^m |\det(B)|^{\frac{1}{2}} |\widehat{v}|_{H^m(\widehat{K})}$$

where C depends on d and m only. Here $H^m(K)$ is the standard Sobolev space.

(ii). Let h_K and $h_{\widehat{K}}$ be the diameters of K and $\widehat{K}$ respectively, and let ρ_K and $\rho_{\widehat{K}}$ be the diameters of the largest circle inscribed in K and $\widehat{K}$ respectively. Show that

$$\|B\| \le \frac{h_K}{\rho_{\widehat{K}}}, \qquad \|B^{-1}\| \le \frac{h_{\widehat{K}}}{\rho_K}.$$